Pergamon

0021–8928(94)00125-1

J. Appl. Maths Mechs, Vol. 58, No. 6, pp. 963–969, 1994

0021–8928/94 $24.00+0.00

ASYMPTOTIC PROPERTIES OF THE SOLUTIONS OF THE EQUATIONS OF MOTION OF GYROSCOPIC SYSTEMS†

A. A. VORONIN

Volzhskii

(*Received* 21 *May* 1993)

The motion of mechanical systems acted upon by gyroscopic and positional forces characterized by a large parameter in the corresponding equations of motion is considered. Periodic solutions of such equations were investigated earlier in [1, 2]. It is proved below that solutions of these equations exist, defined in an interval the length of which is a monotonically increasing unbounded function of the large parameter, and which transfer into the solutions of the corresponding degenerate systems as the large parameter approaches infinity. This function can be specified in more detail if additional assumptions are made regarding the properties of the system and the nature of the forces acting on it.

A similar problem was considered previously in [3] in the case when the forces depend periodically on time. The case of large potential forces was considered in [4] assuming that the degenerate system was stable in the first approximation.

1. Consider a mechanical system with l degrees of freedom, characterized by the following kinetic energy

$$T=\frac{1}{2}\sum_{i,j=1}^{l}a_{ij}(\mathbf{x})\dot{x}_i\dot{x}_j+\sum_{i=1}^{l}a_i(\mathbf{x})\dot{x}_i+h\sum_{i=1}^{2m}b_i(x_1,\ldots,x_{2m})\dot{x}_i+a_o(\mathbf{x})$$

and the generalized forces

$$hQ_i(t,\mathbf{x})\quad(i=1,\ldots,n),\qquad Q_i(t,\mathbf{x})\quad(i=n+1,\ldots,l)$$

Here $\mathbf{x}=(x_1,\ldots,x_l)^{\mathrm{T}}$ are generalized coordinates of the system, a dot above a symbol denotes differentiation with respect to time t, h is a positive large parameter, the symmetric matrix $(a_{ij})_{i,j=1}^{l}$ is positive definite, and $0<2m\geqslant n\geqslant l$.

In the mechanical system considered, large positional forces act along the $x_1,\ldots,x_m$ coordinates and large gyroscopic forces act along the $x_1,\ldots,x_{2m}$ coordinates and are described by the terms with the coefficient h.

Lagrange's equations for the system can be written in the form

$$\begin{aligned}&\frac{d}{dt}(A_{11}\dot{\boldsymbol{\xi}}+A_{12}\dot{\boldsymbol{\eta}}+A_{13}\dot{\boldsymbol{\zeta}})+h(G\boldsymbol{\xi}+\mathbf{Q}^{(1)})=\mathbf{F}_1\\&\frac{d}{dt}(A_{21}\dot{\boldsymbol{\xi}}+A_{22}\dot{\boldsymbol{\eta}}+A_{23}\dot{\boldsymbol{\zeta}})+h\mathbf{Q}^{(2)}=\mathbf{F}_2\\&\frac{d}{dt}(A_{31}\dot{\boldsymbol{\xi}}+A_{32}\dot{\boldsymbol{\eta}}+A_{33}\dot{\boldsymbol{\zeta}})=\mathbf{F}_3\end{aligned}\tag{1.1}$$

†*Prikl. Mat. Mekh.* Vol. 58, No. 6, pp. 22–28, 1994.

Here

$$\boldsymbol{\xi} = (x_1, \ldots, x_{2m})^{\mathrm{T}}, \quad \boldsymbol{\eta} = (x_{2m+1}, \ldots, x_n)^{\mathrm{T}}$$

$$\boldsymbol{\zeta} = (x_{n+1}, \ldots, x_l)^{\mathrm{T}}$$

$$G(\boldsymbol{\xi}) = (g_{ij}(\boldsymbol{\xi}))_{i,j=1}^{2m}, \quad g_{ij} = \frac{\partial b_i}{\partial x_j} - \frac{\partial b_j}{\partial x_i}$$

$$\mathbf{Q}^{(1)} = (Q_1, \ldots, Q_{2m})^{\mathrm{T}}, \quad \mathbf{Q}^{(2)} = (Q_{2m+1}, \ldots, Q_n)^{\mathrm{T}}$$

$$\mathbf{F}_j = \mathbf{F}_j(t, \boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta}, \dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\eta}}, \dot{\boldsymbol{\zeta}}) \quad (j = 1, 2, 3)$$

$$\mathbf{F}_1 \in R^{2m}, \quad \mathbf{F}_2 \in R^{n-2m}, \quad \mathbf{F}_3 \in R^{l-n}$$

The matrices $A_{ij} = A_{ij}(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta})$ are defined by the relation $(A_{ij})_{i,j=1}^{3} = (a_{ij})_{i,j}^{l} = 1$ and have dimensions of A_{11} – $(2m \times 2m)$, A_{22} – $((n - 2m) \times (n - 2m))$, A_{33} – $((l - n) \times (l - n))$, etc. We will assume that $\det G(\boldsymbol{\xi}) \neq 0$ and the matrix $\partial \mathbf{Q}^{(2)}/\partial \boldsymbol{\eta}$ is positive definite for all values of the arguments.

We will convert system (1.1) to a form that can be solved with respect to the leading derivatives. We will carry out the conversions in the form of three successive replacements of variables

$$\dot{\boldsymbol{\zeta}} \to \mathbf{r}, \quad \mathbf{r} = \dot{\boldsymbol{\zeta}} + A_{33}^{-1}(A_{31}\dot{\boldsymbol{\xi}} + A_{32}\dot{\boldsymbol{\eta}})$$

$$\dot{\boldsymbol{\eta}} \to \mathbf{q}, \quad \mathbf{q} = \dot{\boldsymbol{\eta}} + (A'_{22})^{-1} A'_{21}\dot{\boldsymbol{\xi}}$$

$$\dot{\boldsymbol{\xi}} \to \mathbf{p}, \quad P\mathbf{p} = \dot{\boldsymbol{\xi}} + G^{-1}(\mathbf{Q}^{(1)} - A'_{12}(A'_{21})^{-1}\mathbf{Q}^{(2)})$$

where $A'_{ij} = A_{ij} - A_{i3}A_{33}^{-1}A_{3j}$ $(i, j = 1, 2)$, $P = P(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta})$ is a non-degenerate $2m \times 2m$ matrix satisfying the relations

$$P^{\mathrm{T}}(A'_{11} - A'_{12}(A'_{22})^{-1}A'_{21})P = E_{2m}$$

$$P^{\mathrm{T}}GP = -\Gamma = \mathrm{diag}(\gamma_1 J, \ldots, \gamma_m J), \quad J = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix} \tag{1.2}$$

Here and henceforth E_k is the unit $k \times k$-matrix. (We will assume that P and $\gamma_i = \gamma_i(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta})$ are fairly continuous functions for all acceptable $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$.) Multiplying the system obtained on the left by P^{T} and taking (1.2) into account we obtain the following equations

$$\dot{\mathbf{p}} = h\Gamma\mathbf{p} + \mathbf{F}'_1, \quad \dot{\mathbf{q}} = -h(A'_{22})^{-1}\mathbf{Q}^{(2)} + \mathbf{F}'_2, \quad \dot{\mathbf{r}} = \mathbf{F}'_3 \tag{1.3}$$

which, together with the equations of the replacements of variables

$$\begin{gathered} \dot{\boldsymbol{\xi}} = P\mathbf{p} - G^{-1}(\mathbf{Q}^{(1)} - A'_{12}(A'_{22})^{-1}\mathbf{Q}^{(2)}) \\ \dot{\boldsymbol{\eta}} = \mathbf{q} - (A'_{22})^{-1}A_{21}\dot{\boldsymbol{\xi}}, \quad \dot{\boldsymbol{\zeta}} = \mathbf{r} - A_{33}^{-1}(A_{31}\dot{\boldsymbol{\xi}} + A_{32}\dot{\boldsymbol{\eta}}) \end{gathered} \tag{1.4}$$

form a closed system equivalent to Eqs (1.1).

Suppose the equation $\mathbf{Q}^{(2)}(t, \boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta}) = 0$ has the solution $\boldsymbol{\eta} = \boldsymbol{\eta}^0(t, \boldsymbol{\xi}, \boldsymbol{\zeta})$. We will introduce the functions

$$\boldsymbol{\Phi}_{\xi}(t, \boldsymbol{\xi}, \boldsymbol{\zeta}) = -G^{-1}(\boldsymbol{\xi})\mathbf{Q}^{(1)}(t, \boldsymbol{\xi}, \boldsymbol{\eta}^0, \boldsymbol{\zeta})$$

$$\mathbf{q}^0 0(t, \boldsymbol{\xi}, \boldsymbol{\zeta}, \mathbf{r}) = \left(E_{n-2m} + \frac{\partial \boldsymbol{\eta}^0}{\partial \boldsymbol{\xi}} A_{33}^{-1} A_{32} \right)^{-1} \times$$

$$\times \left(((A'_{22})^{-1} A'_{21} - \frac{\partial \boldsymbol{\eta}^0}{\partial \boldsymbol{\zeta}} A_{33}^{-1} \left(A_{31} - A_{32} (A'_{22})^{-1} A'_{21}) + \frac{\partial \boldsymbol{\eta}^0}{\partial \boldsymbol{\xi}} \right) \boldsymbol{\Phi}_\xi + \frac{\partial \boldsymbol{\eta}^0}{\partial \boldsymbol{\zeta}} \mathbf{r} + \frac{\partial \boldsymbol{\eta}^0}{\partial t} \right)$$

$$\boldsymbol{\Phi}_\xi(t, \boldsymbol{\xi}, \boldsymbol{\zeta}, \mathbf{r}) = \mathbf{r} - A_{33}^{-1}(A_{31}\boldsymbol{\Phi}_\xi + A_{32}(\mathbf{q}^0 - (A'_{22})^{-1} A'_{21} \boldsymbol{\Phi}_\xi))$$

$$\boldsymbol{\Phi}_r = \mathbf{F}'_3(t, \boldsymbol{\xi}, \boldsymbol{\eta}^{(0)}, \boldsymbol{\zeta}, \mathbf{0}, \mathbf{q}^0, \mathbf{r})$$

$$(A_{3i} = A_{3i}(\boldsymbol{\xi}, \boldsymbol{\eta}^{(0)}, \boldsymbol{\zeta}), \quad A'_{2j} = A'_{2j}(\boldsymbol{\xi}, \boldsymbol{\eta}^{(0)}, \boldsymbol{\zeta}))$$

$$(i = 1, 2, 3; \quad j = 1, 2)$$

When $h = \infty$ Eqs (1.3) and (1.4) have solutions in which $\mathbf{p} = 0$, $\mathbf{q} = \mathbf{q}^0$, $\boldsymbol{\eta} = \boldsymbol{\eta}^0$, while the variables $\boldsymbol{\xi}$, $\boldsymbol{\zeta}$, $\mathbf{r}$ are defined by the system

$$\dot{\boldsymbol{\xi}} = \boldsymbol{\Phi}_\xi, \quad \dot{\boldsymbol{\zeta}} = \boldsymbol{\Phi}_\zeta, \quad \dot{\mathbf{r}} = \boldsymbol{\Phi}_r \tag{1.5}$$

The mechanical system considered performs a complex motion in which we can distinguish two types of rapid oscillations: mutational (with frequencies $\sim h$), due to the large gyroscopic forces, and oscillations with frequencies $\sim h^{1/2}$ due to the large positional forces. The introduction of the quasi-velocities $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ using (1.4) enables us to distinguish these components in explicit form ("gyroscopic", $\mathbf{p}$ and "positional", $\mathbf{q} - \mathbf{q}^0$).

The degenerate system (1.5) describes the precessional motion with respect to the variables $x_1, \ldots, x_{2m}$ due to the action of the forces hQ_i ($i = 1, \ldots, 2m$), motion along the manifold $\mathbf{q} = \mathbf{q}^0$, $\boldsymbol{\eta} = \boldsymbol{\eta}^0$ due to the action of the forces hQ_1 ($i = 2m + 1, \ldots, n$) with respect to the variables $x_{2m+1}, \ldots, x_n$ and also motion, matched with them, with respect to the variables $x_{n+1}, \ldots, x_l$ described by the last two equations of (1.5).

We will prove that solutions of the equations of motion of the mechanical system considered exist, defined in the interval $0 \leqslant t \leqslant \chi(h)$, where χ is a certain continuous non-negative monotonically increasing unbounded function, and, as $h \to +\infty$ becomes the corresponding solutions of the degenerate system.

Suppose $\boldsymbol{\xi} = \varphi_1(t)$, $\boldsymbol{\zeta} = \varphi_2(t)$, $\mathbf{r} = \varphi_3(t)$ is a certain solution of system (1.5), defined when $0 \leqslant t < +\infty$. We will put $\varphi_4(t)$, $= \boldsymbol{\eta}^0(t, \varphi_1, \varphi_2)$, $\varphi_5(t) = \mathbf{q}^0(t, \varphi_1, \varphi_2, \varphi_3)$. In view of the above assumptions a non-degenerate matrix $S(t)$ exists such that

$$S^{\mathrm{T}} A'_{22}(\varphi_1, \varphi_4, \varphi_2) S = E_{n-2m}$$

$$S^{\mathrm{T}} \frac{\partial \mathbf{Q}^{(2)}(t, \varphi_1, \varphi_4, \varphi_2)}{\partial \boldsymbol{\eta}} S = \operatorname{diag}(\omega_1^2(t), \ldots, \omega_{n-2m}^2(t)) = \Omega(t)$$

We will assume that $S(t)$ and $\omega_i(t)$ ($i = 1, \ldots, n - 2m$) are fairly continuous functions and that the matrix $\Gamma(\varphi_1, \varphi_4, \varphi_2)$ is non-degenerate for all $0 \leqslant t < +\infty$. We will put $\varphi_i^0 = \varphi_i(0)$ ($i = 1, \ldots, 5$).

Theorem 1. For any positive numbers $B_1, \ldots, B_6$ and $\alpha \in (0, \alpha_0)$ ($\alpha_0 = 1/6$) positive constants $C_1, \ldots, C_6$ and H exist as well as a continuous monotonically increasing function $\chi(h)$, unbounded as $h \to +\infty$, such that when $h \geqslant H$ any solution of system (1.3), (1.4) $\mathbf{p}(t, h)$, $\mathbf{q}(t, h)$, $\mathbf{r}(t, h)$, $\boldsymbol{\xi}(t, h)$, $\boldsymbol{\eta}(t, h)$, $\boldsymbol{\zeta}(t, h)$ with initial conditions satisfying the inequalities

$$\|\mathbf{p}(0, h)\| \leqslant B_1 h^{-1}, \quad \|\mathbf{q}(0, h) - \varphi_5^0\| \leqslant B_2 h^{-1}$$

$$\|\mathbf{r}(0, h) - \varphi_3^0\| \leqslant B_3 h^{\alpha - 1}, \quad \|\boldsymbol{\xi}(0, h) - \varphi_1^0\| \leqslant B_4 h^{2\alpha - 2}$$

$$\|\boldsymbol{\eta}(0, h) - \varphi_4^0 - \mathbf{M} h^{-1}\| \leqslant B_5 h^{4\alpha - 2}, \ \|\boldsymbol{\zeta}(0, h) - \varphi_2^0\| \leqslant B_6 h^{\alpha - 1}$$

where $\mathbf{M} = \Omega^{-1}(0)(\mathbf{F}'_2(0, \varphi_1^0, \varphi_4^0, \varphi_2^0, 0, \varphi_5^0, \varphi_3^0) - \dot\varphi_5^0)$, is defined in the section $0 \leq t \leq \chi(h^\alpha)$ and satisfies the following limits in it

$$\|\mathbf{p}(t,h)\| \leq C_1 h^{\alpha-1}, \quad \|\mathbf{q}(t,h) - \varphi_5(t)\| \leq C_2 h^{\alpha-1}$$
$$\|\mathbf{r}(t,h) - \varphi_3(t)\| \leq C_3 h^{2\alpha-1}, \quad \|\boldsymbol{\xi}(t,h) - \varphi_1(t)\| \leq C_4 h^{\alpha-1}$$
$$\|\boldsymbol{\eta}(t,h) - \varphi_4(t)\| \leq C_5 h^{\alpha-1}, \quad \|\boldsymbol{\zeta}(t,h) - \varphi_2(t)\| \leq C_6 h^{2\alpha-1}$$

If there are no large positional forces in the mechanical system considered, then $n = 2m$, and the variables $\mathbf{q}$ and $\boldsymbol{\eta}$ do not occur in Eqs (1.1), while $\varphi_1(t) = \text{const}$. In this case, the degenerate system describes rest with respect to the "gyroscopic" variables $x_1, \ldots, x_{2m}$. It is then necessary to introduce the following changes into Theorem 1

$$\alpha_0 = 1/3, \quad B_1 h^{-1} \to B_1 h^{\alpha-2}, \quad B_3 h^{\alpha-1} \to B_2 h^{\alpha-2}$$
$$B_4 h^{2\alpha-2} \to B_3 h^{\alpha-2}, \quad B_6 h^{\alpha-1} \to B_4 h^{\alpha-2}$$
$$C_3 h^{2\alpha-1} \to C_2 h^{\alpha-1}, \quad C_6 h^{2\alpha-1} \to C_4 h^{\alpha-1}$$

If, moreover, $l = n = 2m$, then there will also be no variables $\mathbf{r}$ and $\boldsymbol{\zeta}$ in system (1.1). All the variables are then "gyroscopic" and we can take the precessional equations as the degenerate system. Choosing $\tau = h^{-1}$ as the independent variable and carrying out transformations in (1.1) similar to those carried out above (similar transformations are carried out in [1]), we obtain equations of the form (1.3) and (1.4) with the replacement $h \to h^2$. It is then necessary to make the following changes in Theorem 1

$$\alpha_0 = 2/3, \quad B_1 h^{-1} \to B_1 h^{-2}, \quad B_4 h^{2\alpha-2} \to B_2 h^{\alpha-4},$$
$$C_1 h^{\alpha-1} \to C_1 h^{\alpha-1}, \quad C_4 h^{\alpha-1} \to C_4 h^{\alpha-2}$$

2. To prove Theorem 1 we will consider the following system of ordinary differential equations

$$\dot{\mathbf{u}} = \mathbf{U}(t, \mathbf{u}, \mathbf{v}), \quad \dot{\mathbf{v}} = h\mathbf{V}_0(t, \mathbf{u}, \mathbf{v}) + \mathbf{V}_1(t, \mathbf{u}, \mathbf{v}) \tag{2.1}$$

Here the dot above a symbol denotes differentiation with respect to t, $\mathbf{u} \in R^n$, $\mathbf{v} \in R^m$ $(n \geq m)$; $\mathbf{U}$, $\mathbf{V}_0$, $\mathbf{V}_1$ are continuously differentiable vector functions of the corresponding dimensions, and $h \gg 1$ is a certain constant.

Suppose the $(m \times (n+m))$-matrix $(\partial\mathbf{V}_0/\partial\mathbf{v}; \partial\mathbf{V}_0/\partial\mathbf{u})$ has a complete rank for all t, $\mathbf{u}$ and $\mathbf{v}$. To fix our ideas we will assume that the matrices $(\partial\mathbf{V}_{0i}/\partial\mathbf{v}_j)$ $(i, j = m - p + 1, \ldots, m)$ and $(\partial\mathbf{V}_{0i}/(\partial\mathbf{u}_j)$ $(i = 1, \ldots, m - p; j = n - m + p + 1, \ldots, n)$ $(0 < p \leq m)$ are non-degenerate for all t, $\mathbf{u}$ and $\mathbf{v}$. We introduce the vectors

$$\mathbf{x} = (\mathbf{x}_1^{\mathrm{T}}, \ldots, \mathbf{x}_4^{\mathrm{T}})^{\mathrm{T}}, \quad \mathbf{x}_1 = (u_1, \ldots, u_{n-m+p})^{\mathrm{T}}, \quad \mathbf{x}_2 = (u_{n-m+p+1}, \ldots, u_n)^{\mathrm{T}}$$
$$\mathbf{x}_3 = (v_1, \ldots, v_{m-p})^{\mathrm{T}}, \quad \mathbf{x}_4 = (v_{m-p+1}, \ldots, v_m)^{\mathrm{T}}$$

and the corresponding vector functions $\mathbf{X}^0 = (0, 0, \mathbf{X}_3^{0\mathrm{T}}, \mathbf{X}_4^{0\mathrm{T}})^{\mathrm{T}}$ and $\mathbf{X} = (\mathbf{X}_1^{\mathrm{T}}, \ldots, \mathbf{X}_4^{\mathrm{T}})^{\mathrm{T}}$ and we rewrite system (2.1) in the form

$$\dot{\mathbf{x}} = h\mathbf{X}^0(t, \mathbf{x}) + \mathbf{X}(t, \mathbf{x}) \tag{2.2}$$

When $h = \infty$ system (2.2) becomes the system

$$\mathbf{X}^0(t, \mathbf{x}) = 0 \tag{2.3}$$

$$\dot{\mathbf{x}}_i = \mathbf{X}_i(t, \mathbf{x}) \quad (i = 1, 2) \tag{2.4}$$

We will call system (2.3), (2.4) a degenerate system. Suppose system (2.3) has an isolated solution $\mathbf{x}_2 = \mathbf{x}_2^0(t, \mathbf{x}_1, \mathbf{x}_3)$, $\mathbf{x}_4 = \mathbf{x}_4^0(t, \mathbf{x}_1, \mathbf{x}_3)$. Substituting these functions into the second equation of (2.3) we obtain the relation

$$\frac{\partial \mathbf{x}_2^0}{\partial t} + \frac{\partial \mathbf{x}_2^0}{\partial \mathbf{x}_1}\mathbf{X}_1 + \frac{\partial \mathbf{x}_2^0}{\partial \mathbf{x}_3}\mathbf{X}_3 = \mathbf{X}_2$$
$$\mathbf{X}_i = \mathbf{X}_i(t, \mathbf{x}_1, \mathbf{x}_2^0, \mathbf{x}_3, \mathbf{x}_4^0) \quad (i = 1, 2, 3)$$

We will assume that this equation (in x_3) has an isolated solution $\mathbf{x}_3 = \mathbf{x}_3^0(t, \mathbf{x}_1)$. Substituting the functions $\mathbf{x}_2^0$, $\mathbf{x}_3^0$ and $\mathbf{x}_4^0$ into the first equation of (2.4) we obtain

$$\dot{\mathbf{x}}_1 = \mathbf{X}_1(t, \mathbf{x}_1, \mathbf{x}_2^0, \mathbf{x}_3^0, \mathbf{x}_4^0) \tag{2.5}$$

Suppose $\mathbf{x}_1 = \varphi_1(t)$ is a certain solution of this system, defined for $0 \leq t < +\infty$. We will put

$$\varphi(t) = (\varphi_1^{\mathrm{T}}(t), \ldots, \varphi_4^{\mathrm{T}}(t))^{\mathrm{T}}, \quad \varphi_3(t) = \mathbf{x}_3^0(t, \varphi_1)$$
$$\varphi_2(t) = \mathbf{x}_2^0(t, \varphi_1, \varphi_3), \quad \varphi_4(t) = \mathbf{x}_4^0(t, \varphi_1, \varphi_3)$$

We make the replacement of variable $\mathbf{x} = \varphi(t) + \boldsymbol{\omega}$ in (2.2) and we separate certain terms in explicit form in the equations obtained. The following system is obtained as a result

$$\dot{\boldsymbol{\omega}} = (hA(t) + B(t))\boldsymbol{\omega} + \mathbf{f}_0^{(0)}(t) + \mathbf{f}_0^{(1)}(t, \boldsymbol{\omega}, h) \tag{2.6}$$

Here

$$A(t) = (A_{ij}(t))_{i,j=1}^4, \quad A_{ij}(t) = \partial \mathbf{X}_i^0(t, \varphi)/\partial \mathbf{x}_j; \quad B(t) = (B_{ij}(t))_{i,j=1}^4, \quad B_{ij}(t) = \partial \mathbf{X}_i(t, \varphi)/\partial \mathbf{x}_j$$
$$\mathbf{f}_0^{(0)} = (0, 0, \mathbf{f}_{03}^{(0)\mathrm{T}}, \mathbf{f}_{04}^{(0)\mathrm{T}})^{\mathrm{T}}, \quad \mathbf{f}_0^{(1)} = (\mathbf{f}_{01}^{(1)\mathrm{T}}, \ldots, \mathbf{f}_{04}^{(1)\mathrm{T}})^{\mathrm{T}}, \quad \mathbf{f}_{0j}^{(0)} = \mathbf{X}_j(t, \varphi) - \dot{\varphi}_j \quad (j = 3, 4)$$

The following estimates hold for the functions $\mathbf{f}_{0j}^{(1)}$ $(j = 1, \ldots, 4)$ as $\boldsymbol{\omega}, h^{-1} \to 0$

$$\|\mathbf{f}_{0j}^{(1)}(t, \boldsymbol{\omega}, h)\| = O(\|\boldsymbol{\omega}\|^2) \quad (j = 1, 2), \quad \|\mathbf{f}_{0j}^{(1)}(t, \boldsymbol{\omega}, h)\| = O(h\|\boldsymbol{\omega}\|^2) \quad (j = 3, 4)$$

We make the replacement of variable $\boldsymbol{\omega} \to \boldsymbol{\omega} + \mathbf{s}h^{-1}$, $\mathbf{s} = (\mathbf{s}_1^{\mathrm{T}}, \ldots, \mathbf{s}_4^{\mathrm{T}})^{\mathrm{T}}$ in system (2.6). The vector components $\mathbf{s}_2$ and $\mathbf{s}_4$ are found as functions of $\mathbf{s}_1$ and $\mathbf{s}_3$ from the system $A\mathbf{s} = \mathbf{f}_0^{(0)}$, while the components $\mathbf{s}_1$ and $\mathbf{s}_3$ are defined by the first two equations of the system $\dot{\mathbf{s}} = B\mathbf{s}$ and the initial condition $\mathbf{s}_1(0) = 0$ after substituting the expressions for $\mathbf{s}_2$ and $\mathbf{s}_4$ into them. (The conditions for these systems to be solvable are the same as for system (2.3), (2.4) with respect to the variables $\mathbf{x}_1, \ldots, \mathbf{x}_4$.)

As a result we obtain the following system

$$\dot{\boldsymbol{\omega}} = (hA(t) + C(t))\boldsymbol{\omega} + \mathbf{f}_1^{(0)}(t, h^{-1}) + \mathbf{f}_1^{(1)}(t, \boldsymbol{\omega}, h) \tag{2.7}$$

where

$$C(t) = (C_{ij}(t))_{i,j=1}^4, \quad C_{ij} = B_{ij} \quad (i = 1, 2; \ j = 1, \ldots, 4)$$
$$C_{ij} = B_{ij} + \partial \mathbf{f}_{0i}^{(1)}(t, \mathbf{s}, h)/\partial \omega_j \quad (i = 3, 4; \ j = 1, \ldots, 4)$$

The following estimates hold for the functions $f_1^{(0)}$ and $f_1^{(1)}$ as $\boldsymbol{\omega}, h^{-1} \to 0$

$$\|\mathbf{f}_{1i}^{(0)}(t, h^{-1})\| = O(h^{-2}), \quad \|\mathbf{f}_{1i}^{(1)}(t, \boldsymbol{\omega}, h)\| = O(h^{-1}\|\boldsymbol{\omega}\| + \|\boldsymbol{\omega}\|^2)$$
$$\|\mathbf{f}_{1j}^{(0)}(t, h^{-1})\| = O(h^{-1}), \quad \|\mathbf{f}_{1j}^{(1)}(t, \boldsymbol{\omega}, h)\| = O(h^{-1}\|\boldsymbol{\omega}\| + h\|\boldsymbol{\omega}\|^2) \quad (i = 1, 2; \ j = 3, 4)$$

We make the replacement of variable $\boldsymbol{\omega} \to \boldsymbol{\omega} + \mathbf{q}h^{-1}$, $\mathbf{q} = (0, \mathbf{q}_2^{\mathrm{T}}, 0, \mathbf{q}_4^{\mathrm{T}})^{\mathrm{T}}$ in system (2.7). The vector components q_2 and q_4 are defined by the system

$$A\mathbf{q} = (0, 0, \mathbf{f}_{13}^{(0)\mathrm{T}}, \mathbf{f}_{14}^{(0)\mathrm{T}})^{\mathrm{T}}.$$

As a result we obtain

$$\begin{aligned} &\dot{\boldsymbol{\omega}} = (hA(t) + D(t))\boldsymbol{\omega} + \mathbf{f}_2^{(0)}(t, h^{-1}) + \mathbf{f}_2^{(1)}(t, \boldsymbol{\omega}, h) \\ &(D(t) = C(t) + \partial \mathbf{f}_1^{(1)}(t, \mathbf{q}, h) / \partial \boldsymbol{\omega}) \end{aligned} \tag{2.8}$$

The estimate for the function $\mathbf{f}_2^{(1)}$ as $\boldsymbol{\omega}, h^{-1} \to 0$ does not change, and for $\mathbf{f}_2^{(0)}$ it has the form

$$\|\mathbf{f}_2^{(0)}(t, h^{-1})\| = O(h^{-2})$$

By virtue of these estimates positive numbers δ and H_1 exist as well as a function $\Phi_0(t)$ such that for all t, $\boldsymbol{\omega}$, $\overline{\boldsymbol{\omega}}$, h satisfying the inequalities $h \geqslant H_1$, the following relations are satisfied

$$\begin{aligned} &\|\mathbf{f}_2^{(0)}(t, h^{-1})\| \leqslant \Phi_0(t) h^{-2}, \quad \|\mathbf{f}_2^{(1)}(t, \boldsymbol{\omega}, h)\| \leqslant \Phi_0(t)(h^{-1}\|\boldsymbol{\omega}\| + h\|\boldsymbol{\omega}\|^2) \\ &\|\mathbf{f}_2^{(1)}(t, \boldsymbol{\omega}, h) - \mathbf{f}_2^{(1)}(t, \overline{\boldsymbol{\omega}}, h)\| \leqslant \Phi_0(t)(h^{-1}\|\boldsymbol{\omega} - \overline{\boldsymbol{\omega}}\| + h(\|\boldsymbol{\omega}\| + \|\overline{\boldsymbol{\omega}}\|)\|\boldsymbol{\omega} - \overline{\boldsymbol{\omega}}\|) \end{aligned} \tag{2.9}$$

We will introduce the set $I = \{(t, s, h){:}\ 0 \leqslant s \leqslant t < +\infty, h \geqslant H_1\}$.

We denote by $W(t, s, h)$ the fundamental matrix of the non-uniform linear system corresponding to (2.8), defined in the set I.

Theorem 2. Suppose the matrix-function $W(t, s, h)$ satisfies the following relation for all $(t, s, h) \in I$

$$\| W(t, s, h) \| \leqslant \Phi_1(t) \tag{2.10}$$

where $\Phi_1(t)$ is a certain continuous function. Then, for any positive numbers B and $\alpha \in (0, \alpha_0)$ $(\alpha_0 = 1/2)$ positive constants C and H exist as well as a continuous non-negative monotonically increasing function $\chi(h)$, unbounded as $h \to +\infty$ such that when $h \geqslant H$ the solution of system (2.8) $\boldsymbol{\omega}(t, h)$ with initial condition which satisfies the inequality

$$\| \boldsymbol{\omega}(0, h) \| \leqslant Bh^{-2} \tag{2.11}$$

is defined in the range $0 \geqslant t \geqslant \chi(h^{\alpha})$ and satisfies the following limit in it

$$\| \boldsymbol{\omega}(t, h) \| \leqslant Ch^{\alpha - 2} \tag{2.12}$$

To prove Theorem 2 we construct a system of integral equations equivalent to the initial problem $\boldsymbol{\omega}(0, h) = \boldsymbol{\omega}_0$ for system (2.8). The existence of a solution of the latter which satisfies conditions (2.11) and (2.12) in the range $0 \geqslant t \geqslant \chi(h^{\alpha})$ can be proved by the method of successive approximations. In this case $\chi = \Phi_2^{-1}$, where $\Phi_2(t)$ is a monotonically increasing non-negative continuous function, unbounded as $t \to +\infty$ satisfying the inequality $\Phi_2(t) \geqslant t\Phi_0(t)\Phi_1(t)$.

We can formulate the following theorem on the basis of Theorem 2 and the replacement of variables made above.

Theorem 3. Suppose that relation (2.10) is satisfied for a certain solution $\mathbf{x} = \varphi(t)$ of the degenerate system (2.3), (2.4). Then, for any positive numbers B and $\alpha \in (0, \alpha_0)$ $(\alpha_0 = 1/2)$ positive constants C and H exist as well as a certain continuous non-negative monotonically increasing function $\chi(h)$, unbounded as $h \to +\infty$ such that for $h \geqslant H$ any solution of system (2.2) $\mathbf{x}(t, h)$ with initial condition satisfying the inequality

$$\| \mathbf{x}(0, h) - \boldsymbol{\varphi}(0) - \mathbf{Q}h^{-1} \| \leqslant Bh^{-2} \tag{2.13}$$

where $\mathbf{Q} = (\mathbf{0}, \mathbf{Q}_2^T, \mathbf{Q}_3^T, \mathbf{Q}_4^T)^T$ is a constant vector, determined using the right-hand side of (2.2) and the initial conditions $\boldsymbol{\varphi}(0)$ and $\dot{\boldsymbol{\varphi}}(0)$, is defined in the interval $0 \leqslant t \leqslant \chi(h^\alpha)$ and, in this interval, satisfies the limit $\|\mathbf{x}(0, h) - \boldsymbol{\varphi}(t)\| \leqslant Ch^{\alpha-1}$.

We can change the scheme of conversions of system (2.6) somewhat in order to simplify the linear part of system (2.8).

Since rank $(\partial \mathbf{V}_0/\partial \mathbf{v}) = p \leqslant m$, we will assume that for all $0 \leqslant t < +\infty$ a real non-degenerate continuously differentiable bounded $(m \times m)$-matrix function $S(t)$ exists which satisfies the relation $S^{-1}A'S = \operatorname{diag}(0, A''_{44})$, where $A' = (A_{ij})$ $(i, j = 3, 4)$. Then, the replacements of variables

$$(\boldsymbol{\omega}_3^T, \boldsymbol{\omega}_4^T)^T \to S(\boldsymbol{\omega}_3^T, \boldsymbol{\omega}_4^T)^T, \quad \boldsymbol{\omega} \to L(t, h^{-1})\dot{\boldsymbol{\omega}} + \mathbf{l}(t, h^{-1})$$

where the matrix L and the vector $\mathbf{l}$ are defined using the right-hand side of (2.6) and satisfy the relations $\det L = 1 + O(h^{-2})$, $\|\mathbf{l}\| = O(h^{-1})$, reduces system (2.6) to a form similar to (2.8) with a linear part in the form of three independent subsystems (corresponding to "slow" and two types of "fast" motions). This simplifies the check of the condition (2.10) but leads to some change in α_0 and the exponents in inequality (2.13).

The equations of motion of gyroscopic systems are a special case of system (2.1). By virtue of the non-degeneracy of the kinetic-energy matrix, and also in view of the oscillating form of the fast motions for these systems, all the conditions of Theorems 2 and 3 are satisfied. The special form of the function $\mathbf{V}_0$ in these systems enables us to equate some (or all) of the vector components $\mathbf{Q}_i$ $(i = 2, 3, 4)$ to zero (by changing α_0 and the exponents in inequality (2.13)).

The corresponding assertions are proved in the same way as Theorem 3.

To specify the form of the mechanical system and the nature of the forces acting on it in more detail the form of the function χ can be refined. For example, for the equations of rotational motion of a satellite-gyrostat under the action of aerodynamic and gravitational moments in a circular orbit [5] we can take $\chi(h) = Th^{\alpha/2}$ $(T = \text{const} > 0)$.

I wish to thank V. V. Sazonov for useful discussions during this research. This research was carried out with partial financial support within the "University of Russia" programme.

REFERENCES

1. VORONIN A. A. and SAZNOV V. V., Periodic motions of gyroscopic systems. *Prikl. Mat. Mekh.* **52,** 5, 719–729, 1988.
2. VORONIN A. A. and SAZONOV V. V., Periodic oscillations of generalized-conservative mechanical systems acted upon by large gyroscopic and potential forces. *Izv. Akad. Nauk. MTT* **6,** 17–19,1992.
3. STRYGIN V. V. and SOBOLEV V. A., *Separation of Motions by the Method of Integral Manifolds.* Nauka, Moscow, 1988.
4. SAZONOV V. V., The dependence of the solutions of the equations of motion of mechanical systems on a large parameter. *Prikl. Mat. Mekh.* **54,** 5, 709–716, 1990.
5. SAZONOV V. V. and VORONIN A. A., Periodic oscillations of a satellite-gyrostat about the centre of mass when acted upon by aerodynamic and gravitational moments. *Kosmich. Issled.* **26,** 4, 492–507, 1988.

Translated by R.C.G.